Holographic Principle bounds on Primordial Black Hole abundances

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Abstract

The generalized Second Law of thermodynamics and the Holographic Principle are combined to obtain the maximum mass of black holes formed inside a static spherical box of size R filled with radiation at initial temperature T_i. The final temperature after the formation of black holes is evaluated, and we show that a critical threshold exists for the radiation to be fully consumed by the process. We next argue that if some form of Holographic Principle holds, upper bounds to the mass density of PBHs formed in the early universe may be obtained. The limits are worked out for inflationary and non-inflationary cosmological models. This method is independent of the known limits based on the background fluxes (from cosmic rays, radiation and other forms of energy) and applies to potentially important epochs of PBH formation, resulting in quite strong constraints to Ω_{pbh}.

1 Introduction

It is a known fact that small primordial black holes (PBHs) are not abundant today because these objects evaporate quickly due to Hawking radiation [1]. Rather strong limits to their relative abundances have been obtained , as discussed in several contributions in Refs. [1-4]. An appraisal of these limits has been recently given in Ref.[2], in which the constraints for several initial PBH masses and formation scenarios have been addressed.

Generally speaking, all these methods are based on comparisons of the integrated contribution of Hawking radiation with some background flux and related arguments, such as deuterium abundance and ^{4}He spallation limit. Although the figures depend on the considered interval of masses, it is fairly general to say that present limits suggest $\Omega_{pbh} < 10^{-6}$ for PBHs with masses $\leq$ the Hawking mass M_{haw} (defined as the mass-scale which is evaporating precisely today). We will here explore another complementary method to limit PBHs abundances based on ideas of entropy bounds.

It is well known that black holes possess some peculiar properties. For instance, they are the most entropic objects for a given mass and radius found in nature. Expressed in units of the Boltzmann constant k_B, a black hole has a huge entropy given by the horizon area as

$$S_{bh}(M) = \frac{A}{4} \sim 10^{77} (M/M_{\odot})^2 \,. \tag{1}$$

It is important to note that the total observed entropy of the universe is just about 10 orders of magnitude greater than a $M \sim M_{\odot}$ black hole, some $S_0 \sim 10^{87}$, contained mainly in the radiation originated in the Big Bang.

The origin and interpretation of this huge entropy is a complicated issue and its complete understanding will establish an important bridge between quantum mechanics, general relativity and thermodynamics. The main problem lies in a statistical interpretation of this entropy. In statistical mechanics, the entropy is associated to the counting of quantum microstates of the system as given by $S \propto \log \Omega$. But which are the quantum microstates of a black hole and how can we count them? These interesting interpretation problems were analyzed in Bekenstein [6], Jacobson [7] and several recent works which address the nature of black hole entropy and attempt a more comprehensive understanding (see [8] for recent reviews).

It is the purpose of this work to obtain conditions for PBH formation using entropy concepts. Next section reviews some general concepts such as the Holographic Principle and the generalization of the second law of thermodynamics, to be needed later. Section 3 discusses the case of a closed box in which PBH formation is induced by external perturbations in order to show how these bounds work. Section 4 presents a brief discussion on the preferred final state. In section 5 non-inflationary cosmological scenarios are addressed and limits to the PBHs abundance derived. Section 6 discusses the same problem in presence of inflation and bounds on the physical temperature for PBHs to form. Conclusions are the subject of Section 7.

2 Entropy bounds and the Generalized Second Law of thermodynamics

The Holographic Principle states that for a given volume V, the state of maximal entropy is proportional to the area A bounded by the volume V. The microscopic entropy S associated with the volume V is bounded by the Bekenstein-Hawking entropy

$$S_{BH} \leq \frac{A}{4} \tag{2}$$

in which the Planck length L_p has been set to unity. Verlinde [9] observed that this bound has to be modified in the cosmological setting with arbitrary dimensions. That work illuminated some interesting connections between entropy and dynamics; and showed that different forms of cosmological entropy bounds may be closely related; namely the Bekenstein-Hawking form (eq.2), the Bekenstein-Verlinde bound $S_{BV} = \frac{2\pi}{n} ER$(with n the number of spatial dimensions) possibly interpreted as a consequence of the quantum Principle of Uncertainty [10] ; and the Hubble bound $S_H = (n-1)\frac{HV}{4G_n}$ (with G_n the generalized Newton constant). At the point in which the Hubble parameter times the radius R satisfy $HR = 1$ the Friedmann-Robertson-Walker equations yield a value of the entropy akin to the Cardy-Verlinde formula [9]. These are strong hints (but not rigorous proof) that the dynamics of the cosmological model has a built-in knowledge of the entropy within its limits.

A second important concept to be used below is the generalized Second Law of thermodynamics, first formulated by Bekenstein in 70s [11]. In a series of gedanken experiments, Bekenstein noted that there were serious problems with the matter/radiation entropy as it was absorbed onto black holes (causing a mass growth of the black hole that absorbs that matter/radiation). However, a series of powerful no-hair theorems granted that black holes are described by very few macroscopic parameters (mass, angular momentum and charge), and after absorption of matter, it is not possible to retrieve the initial conditions and describe the quantum numbers of the absorbed matter incident onto this object. In other words, Bekenstein noted that as black holes absorb matter, the entropy of the universe seems to decrease, since the matter entropy vanished behind the horizon. This was quite problematic, since this *Geroch* process seems to go against the second law of thermodynamics $\Delta S_{matter} > 0$.

In order to recover the second law of thermodynamics, Bekenstein [11] conjectured a Generalized Second Law of Thermodynamics (or GSL), suitable to be applied to black holes and the matter/radiation outside of these objects. According to this conjecture the total entropy of the universe plus N black holes is given by the sum of the matter/radiation entropy plus black hole entropies (which are proportional to their horizon areas). Then, the GSL takes the form

$$S_{total} = S_{m-r} + \frac{1}{4}\sum_{i}^{N} A_i \tag{3}$$

As long as we deal with classical process involving black holes and matter, the total variation of entropy must be positive

$$\Delta S_{total} > 0 \tag{4}$$

The GSL can be used analogously to the well-known use of the second law for ordinary systems. In the section 3 below, we shall use eq.(2) together the GSL for a given box with a well-defined energy and radius to show how to obtain upper limits to the black hole masses formed inside the box. In sections 5 and 6 we use the Holographic Bound, $S_{total} < \frac{A}{4L_{pl}^2}$, conjectured to be valid for weakly or strongly gravitating systems, to repeat the exercise and show the limits for the case of realistic cosmologies.

3 Black hole masses allowed in a closed box

Before considering the case of realistic cosmologies let us begin with a simpler system, namely a finite closed box with thermal radiation in which black hole formation is induced. In the first gedanken experiment this closed ideal box is filled with thermal radiation at initial temperature T_i and no black holes at all. Then, we allow that this system to pass through a finite region with strong metric perturbations. The final state of the experiment is a box containing thermal radiation at some final temperature and black holes with a particular Initial Mass Function (hereafter IMF), the details of the latter depending on the perturbed region. This is known as a Susskind process and it is discussed in [12].

We are not interested here in the intermediate stages, because it is enough to evaluate the initial and final entropies, taking into account the conserved energy, and compare it with the Holographic Bound. Then, we will assume that exists some particular metric perturbations (acting over a finite time) in this box in order to initiate the collapse of some fraction of initial energy in black holes. Let us suppose that this perturbation gives rise to N black holes with same initial masses M_i. By further imposing that $V_{box} >> \sum_i^N {r_g}^3$ (where r_g is the gravitational radius of the black hole), we shall avoid complex interactions between them.

Since we are not modelling the perturbations themselves, we need this condition to assure that the radius and total energy of our boxes are well defined. In practice, we will not explicitly use these constraints, since that we only need to require the background to be asymptotically flat. The complete requirements for PBH formation are then two, enough primordial perturbations and enough entropy for the final state (black holes plus radiation). The condition of having enough perturbations can fix only the functional form of the initial mass function of the PBHs (IMF). The latter condition (enough entropy) completely defines the IMF, determining the lower and upper masses cut-off in the IMF (see below).

The Holographic Principle says that a finite system has a maximal total entropy specified by its radius and energy, and the Bekenstein-Verlinde limit adopts the numerical value

$$S_{BV} = \frac{2\pi}{n} ER \sim 2 \times 10^{38} (M/g)(R/cm) \tag{5}$$

where $n = D - 1$ is the spatial dimension (hereafter set to $n = 3$), M stands for the total mass-energy enclosed in this box and R is its physical radius. The spherical box with radius R, containing thermal radiation at temperature T has an entropy content given by

$$S_{rad}(T) = g_*(T)(TR)^3 \tag{6}$$

and its numerical value is $\sim 2 \times 10^{87}$ for $T = T_0 \sim 3K$ and $R = R_0 \sim 10^{28} {h_0}^{-1} cm$ (the values chosen for the box to mimic the present universe).

We can write the entropy of the box in its final state, with N black holes plus leftover radiation as

$$S_{total} = NS_{bh}(M) + S_{rad}(T) < S_{BV} \tag{7}$$

The total entropy variation is the sum of the entropy variation of the total content of the box and the entropy of the perturbations responsible for the black holes formation. The irreversibility of the total process is expressed as

$$\Delta S_{total} = \Delta S_{box} + \Delta S_{pert} > 0 \tag{8}$$

To calculate the complete outcome of the process we only need to know the entropy and energy in the box, since that the extensive parameters of the box are well-defined.

Provided $S_{box} < S_{BV}$ holds for all times, the formation of the black holes is possible if the difference $[S_{BV} - S_{rad}(T)]$ is positive, because black hole entropies are positive. Then, using the Bekenstein-Verlinde Holographic Bound $S_{BV} = \frac{2\pi}{3}(E_{total}R)$ we obtain

$$\frac{2\pi}{3} ER \geq S_{rad}(T) \tag{9}$$

Considering $E_{total} = \varrho_{total}R^3$, and inserting above, we obtain a constraint on the total density for black hole formation to occur

$$\varrho_{total} \geq \frac{3}{2\pi}\frac{S_{rad}(T)}{R^4} \tag{10}$$

(as we shall show in the next sections, an analogous relation holds for a closed FRW universe when we consider a generalization of the Holographic Principle to this cosmological situation)

Let us show how the formation of PBHs is characterized by the entropy and energy by means of a simple explicit calculation. The initial state of the box will be defined by

$$E_i = \varrho_{rad}(T_i)V_{box} \tag{11}$$

and

$$S_i(T) = g_* {T_i}^3 V_{box} \tag{12}$$

and the final state described by

$$E_f(M_{bh}, T_f) = NM_{bh} + \varrho_{rad}(T_f)V_{box} \tag{13}$$

and

$$S_f(M_{bh}, T_f) = g_* T_f^3 V_{box} + f_0 N {M_{bh}}^2 \tag{14}$$

with $f_0 = 2.5 \times 10^{10}$, the masses are measured in grams and the temperatures in Kelvin degrees. Using the conservation of energy, the radiation temperature is found to fall as

$$T_f = T_i \times \xi(N, M) \tag{15}$$

where we have defined the function $\xi(N, M) = \left[1 - \frac{c^2 NM}{E_i}\right]^{1/4}$. Since the total energy content in the box is conserved, $E(t_f) = E_i$, the eqs.(7),(13) and (14) can be combined into a single expression of the form

$$BM^2 + g_* V_{box} {T_i}^3 \xi^3(N, M) \leq C(E_i/c^2) {V_{box}}^{1/3} \quad (16)$$

with $B \sim 2.5 \times 10^{10} N$, $C \sim 2 \times 10^{38}$ and $E_i = \varrho_0 {T_i}^4 V_{box}$ (with $\varrho_0 \sim 8 \times 10^{-36} gcm^{-3}$).

We seek the maximum values of the formed black hole masses satisfying the Holographic Principle. First, we note that exists an apparent maximum for the black hole mass $M_{max} = (E_i/c^2 N)$. In this case, the final temperature of the radiation is zero. Then, the final state consists of N black holes with this maximum mass, and an extreme cooling of the box happens, $T_f = 0$.

However, for high temperatures, $T_i > T_{cr}$, (holding N, V_{box} and M fixed) there will be an new local maximum for the black hole masses, which clearly satisfies $M_{max} < E_i/c^2 N$. In order to obtain this new maximum, we impose that $S_f(M, T_f) < S_{BV}$ and recall that $E_f(M, T_f) = N M_{bh} + \varrho_{rad}(T_f) V_{box}$. The entropy relation becomes

$$f(N, M) \leq g(T_f(M), V_{box}) \quad (17)$$

where $f(N, M) = (BM^2 - CNM {V_{box}}^{1/3})$ and $g(T_f, V_{box}) = (1.6 \times 10^3 {T_f}^4 {V_{box}}^{4/3} - g_* V_{box} {T_f}^3)$.

On the curve $f(N, M_*) = g(M_*, V_{box})$ we find the maximum value of the mass M_* allowed by the HB. This maximum will be *smaller* than the apparent value $E_i/c^2 N$ if the initial temperature of the box is larger than a critical value T_{cr}, to be evaluated below. The situation is displayed in Figure 1.

When $T_f = T_i$ (i.e. in the limit of $M \to 0$ PBHs), the function $g(T_f, V_{box})$ is positive only for $T_i > 10^{-3} {V_{box}}^{-1/3} K$. As $M \to M_{max}$, the function $g \to 0$, since the final temperature T_f also approaches zero.

We further note that $f(N, M)$ is positive for large black holes and changes signs at smaller values. The global minimum of f is given by $\left(\frac{\partial f}{\partial M}\right)_{M=M_1} = 0$; and leads to $M_1 = \frac{CN}{2B} {V_{box}}^{1/3} \sim 5 \times 10^{27} {V_{box}}^{1/3} g$. The function f adopts at its minimum the value $f(N, M = M_1) = -\frac{C^2 N^2}{4B} {V_{box}}^{2/3}$ and $f(N, M_2) = 0$ occurs for $M_2 = 2M_1$.

An inspection to Fig. 1 shows that the critical temperature (above which the formation of black holes can not exhaust the radiation in the box) must satisfy $2M_1 < \frac{E_i}{c^2 N}$. Solving this inequality we obtain for the critical temperature $T_{cr}(M, N, V_{box})$

$$T_{cr}(M,N,V_{box}) = \left(\frac{2M_1c^2N}{\varrho_0 V_{box}}\right)^{1/4} \sim 5.9\times 10^{15} N^{1/4}(V_{box})^{-1/6} K \tag{18}$$

These calculations lead us to conjecture that if a large number of very large black holes were formed in the primordial universe, they must have exhausted some fraction of the initial temperature of the cosmic radiation due to the energy conservation. This is simple to analyze in a closed finite box, and a similar reasoning for the early universe can be carried over, albeit with several mathematical complications. If we insist that $\varrho_{pbh} << \varrho_{rad}$ then the net effect on the temperature must have been small, although such a sudden cooling might have been sizable, for example, in the hypothesis of large PBH formation as needed for seeds of the supermassive black holes in the center of galaxies. The bottom line here is the belief that the early universe was quite similar (thermodynamically speaking) to a closed box filled initially with radiation, and we may choose the particle horizon $R_{hp}(t)$ as a fiducial value for the effective box that represents it. Therefore, these ideas developed for boxes can be applied to the realistic universe with few modifications in the reasoning (see sections 5 and 6).

4 N small black holes vs. one large black hole

Before jumping to the analysis of cosmological models we can rise the following question when considering perturbations that induce black hole formation in the box. Which situation produces more entropy? (and hence is more likely to occur), N small black holes (with identical masses, say M_*) or one very big black hole (but also satisfying $R_g << V_{box}$)? To address this question let us prepare two boxes with the same initial temperature T_i and same volume. For a meaningful comparison of the entropy variation, we must impose the same *final* temperature. The variation of entropy inside the box for the first case (one black hole formed) is

$$\Delta S_1 = f_0 M^2 - \frac{V_{box}}{T_i}\left[\varrho_{rad}(T_i)\left(1-\frac{1}{\xi(1,M)}\right)+\frac{M}{V_{box}\xi(1,M)}\right] \tag{19}$$

In the second case, the variation of entropy in the box is given by

$$\Delta S_2 = f_0 N {M_*}^2 - \frac{V_{box}}{T_i}\left[\varrho_{rad}(T_i)\left(1 - \frac{1}{\xi(N, M_*)}\right) + \frac{N M_*}{V_{box}\xi(N, M_*)}\right] \tag{20}$$

Since the mass-energy is conserved, $NM_* = M$, and we obtain

$$\Delta S_1 - \Delta S_2 = M^2\left(1 - \frac{1}{N}\right) \tag{21}$$

since $N > 1$ by hypothesis, this difference is positive.

Using eq.(8) we may express the difference of the total entropies between the single large black hole and N black holes as

$$\Delta S^1_{total} - \Delta S^N_{total} = M^2\left(1 - \frac{1}{N}\right) + \Delta S^1_{pert} - \Delta S^N_{pert} \tag{22}$$

Therefore, if the entropy associated to the perturbations (not modelled here) is $\ll$ than $M^2(1 - 1/N)$; or if their difference happens to be small than the latter quantity; the system would prefer to form a single large black hole instead of N small ones. One way to reverse this would be to require a much larger *variation* in the entropy required to form the N black holes than the variation of the entropy involved in the formation of the single large one. Clearly a through evaluation of the unmodelled perturbations is needed to address the outcome on each physically different case.

5 Black hole masses allowed in the early universe: standard (non-inflationary) cosmologies

The considerations of the former sections were useful to understand the basic features of PBHs formation due to its simplicity. This analysis can be extended to the FRW model without drastic modifications. The only difference arises from the cosmological expansion described by the proper FRW dynamics, and the use of a fiducial form (yet to be found unequivocally) of the Holographic Bound. We stress again that there is no rigurous proof of the validity of the HB for the universe as a whole as yet, although we shall use the conservative form of the bound $S_{BH} \leq A/4$ and assume the particle horizon R_{ph} to set the area (and hence the entropy) in which PBHs may form. Thus we shall write S_{ph} for the entropy within the particle horizon.

First we address the important case of subdominant PBHs in the radiation era, arising from the collapse of primordial perturbations ($\varrho_{rad} \gg \varrho_{pbh}$). In this case, the PBH formation does not affect significantly the radiation temperature when they formed (section 3).

The Friedmann equations (without cosmological constant) and entropy density are given by

$$H^2 = \frac{8\pi G}{3}\varrho_{total} - \frac{K}{R^2} \tag{23}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varrho + 3P) \tag{24}$$

By considering a flat universe ($\Omega_0 = 1$), we shall keep the relations between time and redshift as simple as possible. The extension to $\Omega \neq 1$ models and/or alternative functional forms of the Holographic Bound is straightforward to perform.

For this model the entropy inside the horizon is $S_{ph}(t_0) \sim 8 \times 10^{121}$, the particle horizon is given by the integral $R_{ph}(t) = a(t) lim_{\epsilon \to 0} \int_{\epsilon}^{t} \frac{dt}{a(t)} \sim \frac{ct}{(1-n)}$, for $a(t) \propto t^n$. The entropy evolves according to $S_{ph}(t) = S_{ph}(t_0)(t/t_0)^2$, where t_0 is the present time. This dependence of $S_{ph} \propto t^2$ is valid in both the radiation-dominated era or in the matter-dominated era.

The Holographic Bound hypothesis implies that

$$S_{bh}(M) + S_{rad} < A_{ph}(t)/4 \sim 8 \times 10^{121}(t/t_0)^2 \,. \tag{25}$$

The black hole contribution may be written as $S_{bh} = 2.5 \times 10^{40} N \mu^2$ for a delta-type IMF in which all black holes have the same mass and $\mu = (M/10^{15} g)$. If we divide eq.(25) above by the particle horizon volume, the maximal numerical density in black holes (cm^{-3}) is

$$n_{pbh}(t) < \frac{3 \times 10^{-125}}{\pi \mu^2 (t/t_0)^3} \left[8 \times 10^{121} (t/t_0)^2 - S_{rad} \right] \tag{26}$$

which may or may not render relevant limits depending on the difference in brackets.

Since we are considering $\Omega_0 = 1$, then $a(t) = 1/(1+z)$ and $a(t) \propto t^{1/2}$, therefore $t(z) = t_0(1+z)^{-2}$ and $H(z) = H_0(1+z)^2$. Therefore $n_{pbh}(z)$ becomes

$$n_{pbh}(z) < \frac{3 \times 10^{-125}}{\pi \mu^2}(1+z)^6 \left[8 \times 10^{121}(1+z)^{-4} - S_{rad} \right] . \tag{27}$$

Multiplying $n_{pbh}(z)$ by the PBH mass and dividing by the critical density $\varrho_c(z) = \varrho_c(0)[H(z)/H_0]^2$, we finally obtain

$$\Omega_{pbh}(t) < \frac{8\times10^{40}}{\mu}\left[1 - \left(\frac{S_{rad}}{10^{88}}\right)\left(\frac{(1+z)^4}{8\times10^{33}}\right)\right] \tag{28}$$

At a certain redshift z_*, $F(z) = 0$ and PBHs can not be formed. This value is $z_* \sim 3\times10^8$ An inspection to eq.(28) shows that small, but non-zero values of Ω_{pbh} can be obtained if PBHs formed at redshifts z very near z_*, because the prefactor is large for all realistic values of μ. Therefore, in practice it is z_* the redshift that sets a relevant limit. For $z > z_*$ the HB method is unable to provide any useful limit to Ω_{pbh} unless extreme fine-tuning is invoked.

To close this analysis we stress that we have used the solution $a(t) \propto t^{1/2}$, valid when $\varrho_{rad}(t) \gg \varrho_{pbh}(t)$ at any given t in the radiation-dominated era. However, this does not automatically guarantee that subdominant PBHs do not exceed the *entropy* budget. Therefore, we must find the conditions for $\varrho_{rad}(t) \gg \varrho_{pbh}(t)$ and $S_{pbh} \sim S_{rad}$ to be valid simultaneously. Using that $\varrho_{rad}(t) \sim 8.4\times10^5(t/s)^{-2}gcm^{-3}$ and $\varrho_{pbh}(t) = M_{pbh}n_{pbh}(t)$, the radiation dominates the expansion when

$$t(\mu) > 1\times10^{-24}\mu N_{pbh}\, s \tag{29}$$

where N_{pbh} is the total number of PBHs contained within the particle horizon at t. However, the condition $S_{pbh} \sim S_{rad}$ holds only if $N_{pbh}\mu > 4\times10^{47}/\mu$. Combining both we obtain

$$t(\mu) > 4.7\times10^{23}\mu^{-1}\, s\,. \tag{30}$$

For consistency we demand this cosmological time to be within the radiation-dominated era, therefore we must have $t(\mu) < t_D \sim 3\times10^{13}s$, which implies a lower limit to the mass of the PBHs

$$M_{pbh} > 1.4\times10^{26}g \,\sim\, 10^{-7}\, M_\odot \tag{31}$$

The derived bounds are then actually useful to limit PBHs formed inside the radiation-dominated era. Our strongest constraint (always without considering inflation, $\Omega_0 = 1$ and within the radiation-dominated era) is that above $z_* \sim 3\times10^8$ we can not form PBHs, irrespective of the initial mass, provided they satisfy eq.(31). Before this value, the Holographic Bound precludes a PBH formation due to the big entropy of black holes, since the total entropy at disposal is still small. However, since we have not yet considered inflation (which is known to stretch the horizon by several orders of magnitude, among other effects), there may be substantial modifications to be discussed. The bounds within inflationary cosmologies will be the subject of next section.

6 Bounds on PBH masses within inflationary cosmologies

As is well-known, inflation is a brief stage in the evolution of the universe in which the dynamics dictates a huge increase in the horizon size; and thus in the entropy within it. We begin by discussing a constraint on the radius of the particle horizon derived from the same considerations as before (i.e. comparing entropies before and after the formation of N PBHs) subject to the Holographic Bound. We assume that before the PBHs formation $S_{rad}(T(t)) < S_{ph}(t) \sim 8 \times 10^{121} (t/t_0)^2$ holds. At $t > t_f$, we have $S_{rad}(T') + N_{pbh} S_{bh}(\mu) < S_{ph}(t)$, where T' is the lower temperature due to the combined effects of cosmological expansion and the PBH formation (section 3).

It is generally difficult to evaluate the two effects together, but in the approximation in which $\varrho_{pbh} \ll \varrho_{rad}$ at $t = t_f$, we can approximate T' by $T' \sim T[1 - \frac{\varrho_{pbh}}{\varrho_{rad}}]^{1/4} \sim T(1 + \frac{\varrho_{pbh}}{4\varrho_{rad}})$. The relation $N_{pbh} S_{bh}(\mu) + S_{rad}(T(t)) < S_{ph}(t)$ can be inverted to yield the maximum temperature after the N_{pbh} PBH formation

$$T_{max}(\mu, t) \sim \frac{\alpha (t/t_0)^{2/3}}{(R/cm)} \left[1 - \frac{\varrho_{pbh}}{8\varrho_{rad}}\right] \left[1 - \frac{\beta N_{pbh} \mu^2}{(t/t_0)^2}\right]^{1/3} K \tag{32}$$

with $\alpha \sim 4.3 \times 10^{40}$ and $\beta \sim 3 \times 10^{-82}$. Since this quantity must be positive, we must have

$$t(N_{pbh}, \mu) > 1.7 \times 10^{-24} \mu \sqrt{N_{pbh}}\, s\,, \tag{33}$$

to be compared with eq.(30). It may be useful to evaluate S_{ph} in terms of the product $E_{ph}(t)$, the total energy contained inside the particle horizon times its effective radius $R_{ph}(t)$. By doing this, we find the alternative formula for the maximal temperature (we omit hereafter the small correction term $(1 - \frac{\varrho_{pbh}}{8\varrho_{rad}})$)

$$T_{max}(N_{pbh}, \mu) \sim \frac{\gamma}{(R/cm)} \left[\lambda \varrho_{total}(t) (R/cm)^4 - 2.5 N_{pbh} \mu^2\right]^{1/3} K\,, \tag{34}$$

with $\gamma \sim 2 \times 10^{13}$ and $\lambda \sim 8 \times 10^{-2}$. Positivity of this temperature implies that

$$\varrho_{total}(t) > \frac{31 N_{pbh} \mu^2}{(R/cm)^4} g cm^{-3} \tag{35}$$

On the other hand, we know that the physical temperature in the radiation-dominated era behaves as

$$T_{rad} \sim \frac{1.2 \times 10^{29} K}{(R/cm)} \tag{36}$$

PBH formation requires $T_{rad}(t_f) < T_{max}(t_f)$, since that $S_{rad}(t_f)) < S_{ph}(t_f)$, we obtain an additional constraint on the total density for the PBH nucleation to occur, namely

$$\varrho_{total}(t_f) \sim \varrho_{rad}(t_f) < \left[\frac{2 \times 10^{48} + 31 N_{pbh} \mu^2}{(R/cm)^4} \, gcm^{-3} \right] \tag{37}$$

The relation eq.(37) determines a possible range for PBH formation if we know $\varrho_{rad}(t)$ independently. In the radiation-dominated era

$$\varrho_{rad}(t) \sim 8 \times 10^5 (t/s)^{-2} gcm^{-3} \tag{38}$$

and we know that $R(t)$ behaves as $R(t) = R_i a(t) \propto t^{1/2}$, where R_i is the horizon radius at t_i, well before the PBH formation time (we assume that a possible inflationary phase has ended much earlier to avoid complications). R_i corresponds to the beginning of the radiation-dominated era and its value was determined by the end of the inflationary era that connects smoothly with the latter.

Solving the eqs.(37) and (38) we find that the minimum initial radius for PBH formation to happen is

$$R_i > \frac{8 \times 10^{46}}{N_{pbh}^{3/4} \mu^{3/2}} \, cm \tag{39}$$

Clearly, this imposes some conditions to the number of e-folds of the inflation, in order to produce enough entropy given by $R_i \sim L_{pl} \exp[H \Delta t]$. At a given formation time of the PBHs, the number of e-folds enough to solve the usual cosmological problems and to form the PBHs simultaneously must be at least (using eq.39)

$$N_{e-folds} > 190 - \ln(N_{pbh}^{3/4} \mu^{3/2}) \tag{40}$$

The requirement of an inflationary era is necessary in order to get enough entropy in the radiation and thus some room to form Primordial Black Holes. Specific theories with different e-folds will render different abundances in PBHs at the end of the reheating.

6.1 PBHs formed inside the inflationary era

Let us evaluate the surviving PBHs abundances at the end of an inflationary era. We assume that inflation ends with a reheating phase where the inflaton decay strongly heats the cosmic environment. The beginning of the inflation happens when a classical patch of the space-time is filled with some scalar field satisfying $V(\varphi) < M_{pl}^4$, $\partial^\mu\varphi\partial_\mu\varphi < V(\varphi)$. It is assumed that the initial region is larger than the Planck length ($R_0 \sim L_{pl}$) to avoid dealing with a still unknown theory of quantum gravity. In a DeSitter phase the scale factor evolves according to $a(t) = a_i \exp(Ht)$, with $H \sim cte$. The particle horizon in this model will be given by $R_{ph}(t_f) \sim L_{pl}\exp(N_e)$, where t_f corresponds to the end of the inflation. The entropy at that moment $S_{ph}(t_f)$ will be

$$\frac{A_{ph}(t_f)}{4L_{pl}^2} \sim \pi\mathrm{exp}(2N_e) \tag{41}$$

As far as we know, there is no upper bound to N_e, but it must be $\geq$ 67 to get rid of the well-known problems such as homogeneity, horizon, etc. (see [12]). The difference of the Holographic Bound entropy and the radiation entropy (produced by the decay of the inflaton field and of order $\sim 10^{88}$) will be $S_{bh}(\mu) = CN_{pbh}\mu^2 = \pi exp(2N_e) - S_{rad}$, where $C \sim 2\times 10^{40}$. Following the same reasoning of the former sections, we obtain for the PBH density

$$\varrho_{pbh}(t_f) \leq \frac{3\times 10^{15}F(N_e)}{4\pi\mu {L_{pl}}^3 C} \tag{42}$$

with $F(N_e) = \pi\exp(-N_e) - S_{rad}\exp(-3N_e)$.

Today $\varrho_c(t_0) \sim 10^{-29}gcm^{-3}$, but at $t = t_f$ we only know that $\varrho_c(t_f) < {M_{pl}}^4 \sim 10^{93}gcm^{-3}$ (see [14]). A parameter $\eta < 1$ may be introduced to account for these unknown details as $\varrho_c(t_f) = \eta 10^{93}gcm^{-3}$. Using this definition, an upper limit to Ω_{pbh} results, namely

$$\Omega_{pbh}(\mu, N_e) < 6\times 10^{-21}\frac{F(N_e)}{\eta\mu} \tag{43}$$

It is clear that $F(N_e) = 0$ for $N_e^* = \frac{1}{2}ln(\frac{S_{rad}}{\pi}) \sim 100.7$. Then, for N_e below ~ 101 no PBHs can be formed at t_f. The function $F(N_e)$ has a maximum at $N_e^{max} = \frac{1}{2}ln(\frac{3S_{rad}}{\pi}) \sim 101.3$, and the maximal value for $F(N_e)$ is $\sim 2\times 10^{-44}$. Then, the maximal abundance of Ω_{pbh} (at t_f) will be

$$\Omega_{pbh}(t_f) < \frac{1.2\times 10^{-64}}{\eta\mu} \tag{44}$$

The volume of the horizon after t_f will be even larger than $\sim {L_{pl}}^3 \exp(3N_e)$ and Ω_{pbh} drops even more in these conditions. There is no hope of producing a sizeable Ω_{pbh} of PBHs formed inside the inflationary era. The reasons for this huge PBH dilution are similar to that found in Guth's analysis of the magnetic monopoles abundances [13]. The huge injection of entropy (and its associated expansion) dilutes the magnetic monopole abundances to negligible values $\Omega_{mm} \sim 10^{-81}$ or less. For PBHs, the effect is analogous, but as these objects are extremely entropic ($C \sim 10^{40}$), we obtain the bounds of eq.(44) which are equally strong for $\sim 1\, M_\odot$ black holes. Nevertheless, the dilution is always very strong and states that PBHs do not contribute to Ω_{total} if formed within the inflation, as expected.

6.2 PBHs formed after an inflationary era

The inflation of R_{ph} proved to be lethal for PBHs formed inside the inflation, but the former certainly created a favorable setting after t_f for any process that may induce PBH formation. We can evaluate the maximal PBH abundances at $t > t_f$ as follows. We will simplify the model by assuming that the inflationary phase lasts $\Delta t \leq t_f$, and from $t_f = \Gamma \times 10^{-35} s$ on (with $1 < \Gamma < 10^3$ being a model-dependent parameter including the unknown details of the epoch) the universe entered the radiation-dominated era Then $a(t) \propto \exp(t)$ for $t < t_f$, and after this phase, the decay of the inflaton field reheats the background as the universe enters the radiation-dominated era in which $a(t) \propto t^{1/2}$. The numerical details of this transition are not important for our estimates, as long as it occurs immediately after t_f. The continuity of the horizon particle allows us write $R_{ph}(t) = \frac{ct_f}{(1-n)}(t/t_f)$. This value is adopted as the initial R_i of the radiation-dominated era $R_{ph}(t_f) \equiv R_i = \frac{ct_f}{(1-n)} = L_{pl} \exp(N_e)$.

Therefore we have $R_{ph}(t) \sim L_{pl} \exp(N_e)(t/t_f)$ for $t > t_f$, and the entropy evolves as

$$S_{ph}(t) = \pi \exp(2N_e)(t/t_f)^2 \tag{45}$$

Following the same reasoning as above, we write now for Ω_{pbh}

$$\Omega_{pbh}(\mu, N_e, t) < 9.4 \times 10^{-4} \frac{\Gamma^2}{\mu}(t/t_f) \times G(N_e, t) \times \exp\left(-N_e\right) \tag{46}$$

where

$$G(N_e, t) = 1 - \frac{S_{rad}}{\pi (t/t_f)^2} \exp(-2N_e) \tag{47}$$

valid only for $1 < (t/t_f) < (t_D/t_f) \sim (3/\Gamma)10^{48}$, that is, from the end of inflation till the beginning of the matter-dominated era.

We can proceed to evaluate the maximal abundances in PBHs formed at a time t_{form}. Most of the models that describe PBH formation from primordial fluctuations (see [15]), estimate that the PBH mass at its formation, t_{form}, is a fraction $\beta \leq 1$ of the particle horizon. Then, it is reasonable to put

$$\mu(t = t_{form}) \sim \beta(M_{hor}(t_{form})/M_{haw}) \sim 9\beta \times 10^{22}(t_{form}/s) \quad (48)$$

Furthermore, as stated above the initial conditions for inflation are $V(\varphi) >> \partial^\nu\varphi\partial_\nu\varphi$ and $V(\varphi) < M_{pl}^4$. The last one says that the spacetime is classical. We do not know exactly the initial energy density stored by the inflaton field, but it is generally assumed that $V(\varphi) = \eta M_{pl}^4$, with $\eta < 1$. Using the Friedmann equation $H_0^2 \propto V(\varphi)$, $H_0 t_f = N_e$ and recalling the definition of t_f we identify $t_f = \Gamma \times 10^{-35}s \sim 2.2 \times 10^{-44}\frac{N_e}{\eta^{1/2}}s$. Substituting μ and Γ into eq.(46) (evaluated at $t = t_{form}$) yields

$$\Omega_{pbh}(N_e, t = t_{form}) < \frac{2.1 \times N_e}{\beta\eta^{1/2}} \times G(N_e, t_{form}) \times 10^{-\frac{43}{100}N_e} \quad (49)$$

The PBH abundance is zero when the function $G(N_e, t_{form})$ vanishes, this happens at

$$t_* \sim 0.9 N_e \eta^{-1/2} \exp(-N_e)\, s \quad (50)$$

All those inflationary models with a large number of e-folds N_e, will not form PBHs at times earlier than t_*. After $t_*(N_e, \eta)$, the difference between the total entropy (given by the GSL) and the Holographic Bound will allow the PBH formation by collapse of large fluctuations. Other PBH formation mechanism(s) allowing masses much below M_{hor} will suffer milder restrictions, yet to be studied.

7 Conclusions

After analyzing the formation of PBHs inside boxes and possible extensions to realistic cosmological models, we conclude that if the Holographic Bound holds, we can use it to obtain constraints on the black hole formation and other features (see the reviews in [16] and [17] for complete discussions of the Holographic Principle). The usefulness of this method is rooted on the huge value of the black hole entropy, which poses limitations to uncontrolled formation of PBHs. Furthermore, since the early universe was very small (compared to the present universe), the total entropy within it was much smaller than today and useful limits can be extracted from some upper bound to the total entropy. This method can be applied to constrain the epochs and masses of PBH nucleation and new requirements on the early conditions obtained. It is not a feature of a particular form of the Holographic Principle, but rather a general feature that can be implemented for any given form of the latter, and which may prove restrictive for their abundances in several cases. We obtained useful bounds for standard (non-inflationary) and inflationary cosmological models. For the former a threshold redshift z_* quenching the formation of PBHs has been found; while the limits are strong for the latter in all cases. The minimal set of assumptions made for inflationary cosmologies suggests that this low abundance of PBHs is a quite generic feature of the models.

All these cosmological applications made use of the expressions S_{BH} and S_{BV}, without considering, for example, the influence of the equation of state. We may rederive all these results again, considering any generalization of them, such as the proposal made by Youm [18]. The *growth* of PBHs has been also dismissed in all this work. However, it is known that the black holes formed in the very early universe can grow at expenses of the radiation around them (see [19]) and [20]). Note that as PBHs grow with time, the total entropy of the universe grows proportional to their masses. A simplified analysis of the problem of PBHs growth has been presented recently by the authors [21] using model-independent arguments. In all the latter analysis it has been found that these PBHs (almost) did not grow at all, and therefore, the maximum of the entropy was achieved practically at the instant of their formation. In spite that we expect small modifications to the present results when the growth of PBHs is included, a detailed analysis is needed to confirm this expectation.

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Figure captions.

Fig. 1. Graphical solution of the formation of PBHs at the expense of background radiation within a closed box. The curves $g(T_1, V_{box})$ and $g(T_2, V_{box})$ refer to two different cases. In the first, $T_1 >> T_2$ and the formation of the black holes does not exhaust the radiation within the box (upper solid curve). In the second case (lower solid curve), the formation of black holes exhausts the ambient radiation completely. As discussed in the text, the Holographic Bound is more restrictive than the total content of energy to set the maximal mass for individual black holes; unless the conditions in the box are such that $M_{max} < M_2$ (with M_2 the zero of the function $f(N, M)$), since in the latter situation the energy constraint is stronger than the Holographic Bound to evaluate the PBH formation.

Fig. 2. Primordial black hole abundance limits within inflationary models. As discussed in the text,$\Omega_{pbh} \equiv 0$ before t_*. Note that even for PBHs formed at asymptotic times (that is, when $G \to 1$) the allowed contribution to Ω_{pbh} is very small because of the suppression by N_e. Physically the suppression reflects the fact that the volume grows much more than the entropy inside the particle horizon.